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APPROXIMATIONS TO THE RELIABILITY

OF PHASED MISSIONS

by

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A system whose configuration (block diagram or fault tree) changes during consecutive time periods (phases) performs a "phased mission." Recently, Esary and Ziehms have shown that any multi-phase mission can be transformed into an equivalent, synthetic, single-phase system, and thus that the phased mission problem can in principle be solved by standard reliability methods. We employ these ideas here to study (over)

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approximations to mission reliability and to develop an algorithm which may be of practical interest. In addition, we extend the reliability calculus of Rubinstein, and Esary and Hayne, based on an approximate hazard transform, to phased missions, and we show how this extended calculus can be used in situations where phases are not of known fixed duration.

1. INTRODUCTION

The technological development of the last two decades, particularly in the areas of space flight, nuclear power generation, and weapons systems, has forced reliability analysts to consider systems whose configuration change over time. "Phased missions" have received attention in the basic papers of Rubin [5] and Weisberg and Schmidt [9] which present computational procedures to approximately predict mission reliability and crew safety for manned spacecraft, and in the United States Navy reliability manual NAVORD OD 29304 Revision A [8].

Recently, Esary and Ziehms [4] have investigated a phased mission of the following form:

> A system consists of several independently performing components each of which functions continuously in time until failure occurs, and remains failed thereafter; repair or replacement is not possible. The system performs a mission which is divided into consecutive time periods, or phases, of known duration. The system configuration, defined as a subset of the components and their functional organization, changes from phase to phase. As is the case with individual components, only two states of the system are recognized, functioning or failed. The mission is successful if the system functions throughout all phases.

Their main result is that any multi-phase mission of this type can be transformed into an equivalent, synthetic, single-phase system, and thus that the phased mission problem can be solved in principle by standard reliability methods. They point out, however, that a direct implementation of their transformation could be frustrated by a large number of components in the equivalent system.

In this paper we employ the ideas of Esary and Ziehms to study some approximations to mission reliability and to develop an algorithm which may be of practical interest. In addition, we extend the reliability calculus of Rubinstein [6, 7] and Esary and Hayne [1], based on an approximate hazard transform, to phased missions, and demonstrate how the extended calculus can be used in situations where phases are not of known fixed duration.

2. PROBLEM FORMULATION AND PREVIOUS RESULTS

Suppose that the system under consideration has n components, labelled $C_1,\ldots,C_n$, with independent times to failure $T_1,\ldots,T_n$. For all times $t \geq 0$, define the <u>performance state indicator vector</u> of the set of components $\underset{\sim}{X}(t) = (X_1(t),\ldots,X_n(t))$ by $X_k(t) = 1$ iff $T_k > t$, and $X_k(t) = 0$ otherwise, $k = 1,\ldots,n$. Assume that the mission is divided into m phases, and that phase j starts at time t_{j-1} and ends at time t_j, $j = 1,\ldots,m$, with $t_0 = 0$. Finally, let ϕ_j be the structure function which describes the configuration (assumed to be coherent) of the system in phase j, $j = 1,\ldots,m$. Then the event that the mission is successful is $\{\phi_1[\underset{\sim}{X}(t_1)] = 1,\ldots,\phi_m[\underset{\sim}{X}(t_m)] = 1\}$, and the mission reliability p can be expressed as

(1) $$p = E \prod_{j=1}^{m} \phi_j[\underset{\sim}{X}(t_j)].$$

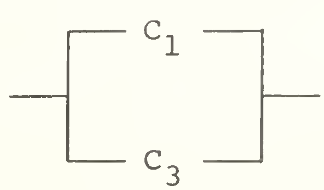
To obtain an equivalent single-phase system, pseudo-components C_{kj} are introduced whose reliabilities are the conditional phase reliabilities of the original components. Formally, for $k = 1,\ldots,n$ and $j = 1,\ldots,m$, the performance state indicator variable U_{kj} of pseudo-component C_{kj} has the distribution

(2)
$$P[U_{k1} = 1] = P[X_k(t_1) = 1],$$
$$P[U_{kj} = 1] = P[X_k(t_j) = 1 \mid X_k(t_{j-1}) = 1], \qquad j \neq 1.$$

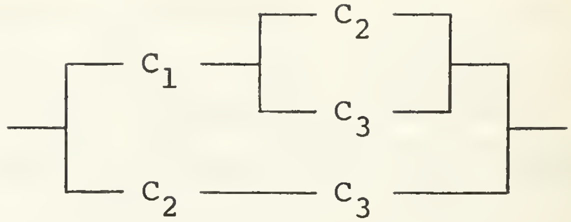
The <u>transformation</u> is accomplished by replacing, in the configuration for phase j, component C_k by a series system in which the pseudo-components $C_{k1},\ldots,C_{kj}$ perform independently with the probabilities of functioning given in (2), and by regarding the transformed phase configurations as subsystems operating in series. As an illustration, consider the following example.

<u>Example 1</u>. A system with three components performs a three-phased mission whose phase configurations can be represented by the block diagrams

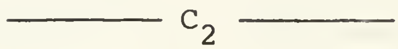
- in phase 1:



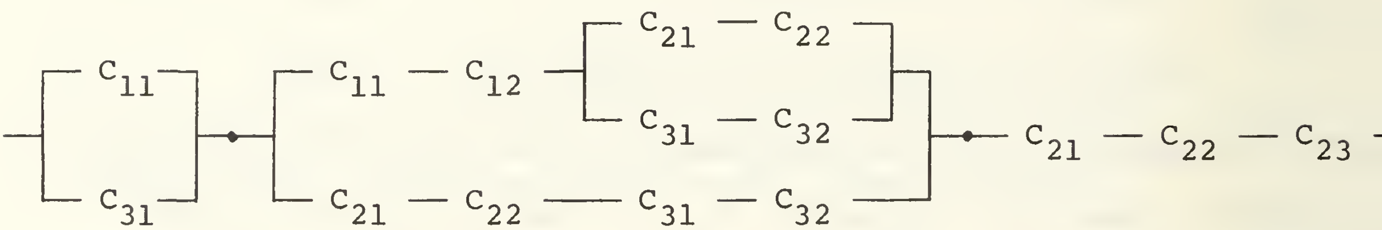
- in phase 2:



- in phase 3:



Then the equivalent system has the block diagram



□

The reliability of the equivalent system is

$$(3) \qquad p = E \prod_{j=1}^{m} \phi_j(\underset{\sim}{U}^{(1)}\underset{\sim}{U}^{(2)}\cdots\underset{\sim}{U}^{(j)}),$$

where $\underset{\sim}{U}^{(i)} = (U_{1i},\ldots,U_{ni})$ and $\underset{\sim}{U}^{(i)}\underset{\sim}{U}^{(\ell)} = (U_{1i}U_{1\ell},\ldots,U_{ni}U_{n\ell})$. The value of p as given by (3) agrees with the value of p as given by (1) [4, Theorem 3.1], and thus the ordinary reliability of the equivalent system whose components perform independently is the same as the reliability of the orginal system for its phased mission.

3. SOME BOUNDS ON MISSION RELIABILITY

An obvious first approach to approximating mission reliability--discussed in [4] and repeated here for the sake of completeness--is to compute the reliability of each phase configuration separately and then to multiply the results together. There are at least two choices of component reliabilities to use in doing this: the component <u>conditional phase reliabilities</u>

$$
(4)\qquad
\begin{aligned}
\pi_{k1} &= P[X_k(t_1) = 1] \\
\pi_{kj} &= P[X_k(t_j) = 1 \mid X_k(t_{j-1}) = 1], \qquad j = 2,\ldots,m,
\end{aligned}
$$

which are the reliabilities of the pseudo-components in the equivalent system, or the component (unconditional) <u>reliabilities</u> through each phase

$$
(5)\qquad \rho_{kj} = P[X_k(t_j) = 1] = \prod_{i=1}^{j} \pi_{ki}, \qquad j = 1,\ldots,m,
$$

$k = 1,\ldots,n$. The first choice leads to approximating mission reliability by

$$
(6)\qquad \pi_{PRF} = \prod_{j=1}^{m} h_j(\pi_{1j},\ldots,\pi_{nj}),
$$

and the second choice to approximating mission reliability by

$$
(7)\qquad \rho_{PRF} = \prod_{j=1}^{m} h_j(\rho_{1j},\ldots,\rho_{nj}),
$$

where in both cases h_j, $j = 1,\ldots m$, are the reliability functions for the phase configurations. (The <u>reliability function</u> of a system with structure function ϕ is defined by $h(p_1,\ldots,p_n) = E\,\phi\,(X_1,\ldots,X_n)$, where $X_1,\ldots,X_n$ are independent Bernoulli random variables with $P[X_k = 1] = p_k$, $k = 1,\ldots,n$.) The subscript PRF in (6) and (7) is meant to indicate that these approximations are based on <u>p</u>hase <u>r</u>eliability <u>f</u>unctions.

It has been shown using (3) [4, Remark 4.1] that (6) gives an optimistic result and that (7) gives a conservative result, i.e. that for p as given by (1) or (3),

$$\rho_{PRF} \leq p \leq \pi_{PRF} . \tag{8}$$

The above approximations can be employed only when the reliability functions of all m phases are known. Although to compute them is considerably easier than to compute the overall reliability function for the equivalent system, it may in practical problems still be a formidable task. We will therefore now discuss an approach which avoids these difficulties.

For coherent single-phase systems with independent components, Esary and Proschan [2] have established two bounds on system reliability which do not involve the reliability function: the <u>minimal path upper bound</u> and the <u>minimal cut lower bound</u>. These bounds, when applied to each phase separately, can be used to approximate mission reliability in the multi-phase case. Let h_{UBj} and h_{LBj} denote the minimal path upper bound and the minimal cut lower bound, respectively, for phase configuration j, $j = 1,\ldots,m$. Using basically the same approach as before, and choosing as component reliabilities the conditional phase reliabilities π_{kj} in one case and the (unconditional) reliabilities ρ_{kj} in the other, we obtain the approximations

$$\pi_{PUB} = \prod_{j=1}^{m} h_{UBj}(\pi_{1j},\ldots,\pi_{nj}) \tag{9}$$

and

(10) $$\rho_{PLB} = \prod_{j=1}^{m} h_{LBj}(\rho_{1j},\ldots,\rho_{nj}),$$

where the subscripts are to indicate that these approximations are based, respectively, on phase upper bounds and phase lower bounds. Since the phase configurations are coherent by assumption, hence $h_{LBj} \le h_j \le h_{UBj}$, $j = 1,\ldots,m$; it thus follows from (6) and (9) that

(11) $$\pi_{PRF} \le \pi_{PUB},$$

and from (7) and (10) that

(12) $$\rho_{PLB} \le \rho_{PRF}.$$

From (8), (11), and (12) we can conclude that (a) is an upper bound on mission reliability, and (10) is a lower bound on mission reliability.

4. CUT CANCELLATION AND FURTHER BOUNDS

Rubin, Weisberg and Schmidt used a method to simplify the sequence of phase configurations prior to beginning reliability calculations which has become known as "cut cancellation." Cut cancellation does not affect mission reliability [4, Remark 4.2] and can be summarized in the following rule:

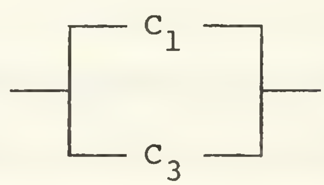
> A minimal cut set in a phase can be cancelled, i.e. omitted from the list of minimal cut sets for that phase, if it contains a minimal cut set of a later phase.

The next example illustrates how cut cancellation works.

Example 2. Consider the mission of Example 1. The minimal cut sets are

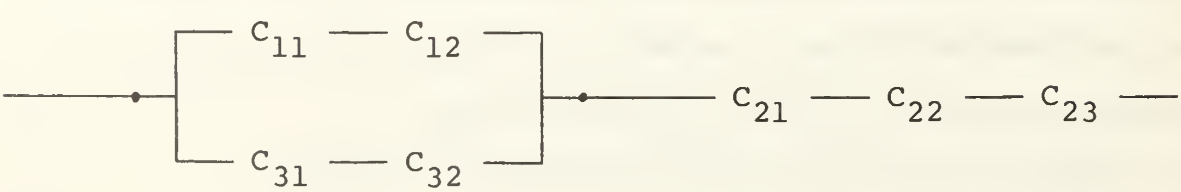
- in phase 1: $\{C_1, C_3\}$
- in phase 2: $\{C_1, C_2\}$, $\{C_1, C_3\}$, $\{C_2, C_3\}$
- in phase 3: $\{C_2\}$

The phase 1 cut set $\{C_1, C_3\}$ contains the phase 2 cut set $\{C_1, C_3\}$, and thus can be cancelled in phase 1, leaving a configuration which can never fail. Both the phase 2 cut sets $\{C_1, C_2\}$ and $\{C_2, C_3\}$ contain the phase 3 cut set $\{C_2\}$, so they can be cancelled in phase 2. After cut cancellation, the simplified phase configurations can be represented by the block diagrams

- in phase 1:
- in phase 2: 

- in phase 3: — C_2 —

After cancellation, the transformation can be applied to obtain the equivalent system with the block diagram



which is considerably simpler than the equivalent system of Example 1, but has the same reliability. □

The methods of approximating mission reliability described in the previous section can also be employed after cut cancellation has been performed. Denoting the reliability functions of the simplified phase configurations by $\bar{h}_j$, $j = 1,\dots,m$, the approximations corresponding to π_{PRF} and ρ_{PRF} are

$$(13) \qquad \pi_{PRF-CC} = \prod_{j=1}^{m} \bar{h}_j(\pi_{1j},\dots,\pi_{nj})$$

and

$$(14) \qquad \rho_{PRF-CC} = \prod_{j=1}^{m} \bar{h}_j(\rho_{1j},\dots,\rho_{nj}),$$

respectively, where the added subscript CC indicates that cut cancellation has been performed. Similarly, denoting by $\bar{h}_{UBj}$ and $\bar{h}_{LBj}$ the minimal path upper bound and the minimal cut lower bound, respectively, for the simplified configuration of phase j, $j = 1,\dots,m$, we obtain the approximations

$$(15) \qquad \pi_{PUB-CC} = \prod_{j=1}^{m} \bar{h}_{UBj}(\pi_{1j},\dots,\pi_{nj})$$

and

$$(16) \qquad \rho_{PLB-CC} = \prod_{j=1}^{m} \bar{h}_{LBj}(\rho_{1j},\dots,\rho_{nj}).$$

To show that these four approximations are bounds on mission reliability, we observe first that since the simplified phase configurations are coherent, hence $\bar{h}_{LBj} \le \bar{h}_j \le \bar{h}_{UBj}$, $j = 1,\dots,m$. It follows from (13) and (15) that

(17) $$\pi_{PRF-CC} \leq \pi_{PUB-CC},$$

and from (14) and (16) that

(18) $$\rho_{PLB-CC} \leq \rho_{PRF-CC}.$$

Further, since the phase reliability functions are not less after cut cancellation than before, i.e. $h_j \leq \bar{h}_j$, $j = 1,\ldots,m$, then

(19) $$\pi_{PRF} \leq \pi_{PRF-CC}$$

follows from (6) and (13), and

(20) $$\rho_{PRF} \leq \rho_{PRF-CC}$$

follows from (7) and (14), where the latter inequality is noted here for further reference only. From (19) and (8) we conclude that π_{PRF-CC} and π_{PUB-CC} are in fact upper bounds on mission reliability.

To establish that ρ_{PRF-CC} and ρ_{PLB-CC} are lower bounds, we need the following remark.

Remark 1. Let $\bar{\phi}_j$ be the structure function of the simplified configuration of phase j, $j = 1,\ldots,m$, and let U_{kj}, $k = 1,\ldots,n$, $j = 1,\ldots,m$, be the indicator variables of the pseudo-components in the equivalent system. Then

$$\prod_{j=1}^{m} E\bar{\phi}_j(\underset{\sim}{U}^{(1)}\underset{\sim}{U}^{(2)}\ldots\underset{\sim}{U}^{(j)}) \leq E\prod_{j=1}^{m} \bar{\phi}_j(\underset{\sim}{U}^{(1)}\underset{\sim}{U}^{(2)}\ldots\underset{\sim}{U}^{(j)}).$$

Proof. The proof uses standard properties of associated random variables which are discussed in [3].

The simplified phase configurations are coherent, and hence the structure functions ϕ_j^-, $j = 1,\ldots,m$, are non-decreasing. The Bernoulli random variables U_{kj}, $k = 1,\ldots,n$, $j = 1,\ldots,m$, are independent. Therefore $\phi_j^-(\underset{\sim}{U}^{(1)}\underset{\sim}{U}^{(2)}\ldots\underset{\sim}{U}^{(j)})$, $j = 1,\ldots,m$, are associated Bernoulli random variables for which the assertion of the remark holds. □

Using (2), (5), and (14) we obtain $\rho_{PRF-CC} = \prod_{j=1}^{m} E\phi_j^-(\underset{\sim}{U}^{(1)}\underset{\sim}{U}^{(2)}\ldots\underset{\sim}{U}^{(j)})$; since cut cancellation does not affect mission reliability, (3) can be written as $p = E\prod_{j=1}^{m} \phi_j^-(\underset{\sim}{U}^{(1)}\underset{\sim}{U}^{(2)}\ldots\underset{\sim}{U}^{(j)})$. Application of Remark 1 then yields the inequality

$$\rho_{PRF-CC} \leq p, \tag{21}$$

which together with (20) establishes the desired results.

5. COMPARISONS OF THE BOUNDS

The magnitudes of the bounds on mission reliability presented in the previous sections, and of the mission reliability itself, can be ordered. This ordering is displayed in Figure 1 where the superscripts refer to the defining equations and inequalities which are summarized.

No general inequalities can be established between π_{PRF-CC} and π_{PUB}, and between ρ_{PLB-CC} and ρ_{PRF}. In the case of the two upper bounds, cut cancellation on one hand and the use of phase upper bounds instead of phase reliability functions on

$$
\begin{array}{ccccccccccccc}
 & \le^{22} & \rho_{PLB\text{-}CC}{}^{16} & \le^{18} & & & & & \le^{19} & \pi_{PRF\text{-}CC}{}^{13} & \le^{17} & \pi_{PUB\text{-}CC}{}^{15} \\
\rho_{PLB}{}^{10} & & & & \rho_{PRF\text{-}CC}{}^{14} & \le^{21} & p^{1,3} & \le^{8} & \pi_{PRF}{}^{6} & & & \\
 & \le^{12} & \rho_{PRF}{}^{7} & \le^{20} & & & & & \le^{11} & \pi_{PUB}{}^{9} & &
\end{array}
$$

Figure 1. Qualitative comparison of the bounds on mission reliability

(The superscripts refer to the defining equations and inequalities.)

the other hand both tend to increase the apparent phase reliabilities, the amount of increase depending on the structure of the mission as well as on the component reliabilities. In the case of the two lower bounds, ρ_{PLB-CC} tends to be greater than ρ_{PRF} because of cut cancellation, but also smaller because of the use of phase lower bounds instead of phase reliability functions. Again, both the structure of the mission and the component reliabilities determine which of them is greater in a particular case.

The inequality

$$\rho_{PLB} \leq \rho_{PLB-CC} \tag{22}$$

has not yet been established formally, but is an obvious consequence of (10), (16), and the fact that $h_{LBj} \leq \bar{h}_{LBj}$, $j = 1,\ldots,m$. A similar inequality between the upper bounds π_{PUB} and π_{PUB-CC}, however, does not exist, because it is not necessarily true that $h_{UBj} \leq \bar{h}_{UBj}$. Since this may not be intuitively obvious, we give the following illustration.

Example 3. For the mission of Example 1, the minimal path upper bound for phase 2 is $h_{UB2}(\pi_{12},\pi_{22},\pi_{32}) = \pi_{12}\pi_{22} \vee \pi_{12}\pi_{32} \vee \pi_{22}\pi_{32}$ before cut cancellation, and $\bar{h}_{UB2}(\pi_{12},\pi_{22},\pi_{32}) = \pi_{12} \vee \pi_{32}$ after cut cancellation. Assuming that $\pi_{12} = \pi_{22} = \pi_{32} = \pi$, then $h_{UB2}(\pi) = \pi^2(3-3\pi^2+\pi^4)$ and $\bar{h}_{UB2}(\pi) = \pi(2-\pi)$. For $0 < \pi \leq .8$, $h_{UB2}(\pi) < \bar{h}_{UB2}(\pi)$, and for $.9 \leq \pi < 1$, $h_{UB2}(\pi) > \bar{h}_{UB2}(\pi)$. □

It is also possible to compare the bounds with respect to the computational effort required to compute them. In general, less effort is required to compute the m phase reliability functions separately than to compute one reliability function for the equivalent system; phase bounds are easier to compute than phase reliability functions; and cut cancellation simplifies all reliability calculations, although it requires computational effort itself. The diagram below is an attempt to summarize these observations. Its comparisons may not hold in all cases, but do indicate what is usually true. The symbol $\leftarrow$ stands for "requires less computational effort than."

$$\begin{array}{lll} \pi_{PUB-CC} \leftarrow \pi_{PUB} \leftarrow \pi_{PRF-CC} \leftarrow \pi_{PRF} & & \\ & \leftarrow & p \\ \rho_{PLB-CC} \leftarrow \rho_{PLB} \leftarrow \rho_{PRF-CC} \leftarrow \rho_{PRF} & & \end{array}$$

6. AN ALGORITHM FOR THE "BEST" LOWER BOUND

Trying to select the best bound from those presented here is a problem whose solution depends on the circumstances of each particular application and cannot be given in general. If one is interested in a conservative rather than an optimistic approximation, and if the system to be analyzed has components with uniformly high conditional reliabilities in all phases, then the qualitative comparisons of the previous section and numerical results suggest that ρ_{PLB-CC} is a good choice. Since these conditions are frequently encountered, an algorithm for computing ρ_{PLB-CC} is given below. Inputs to this algorithm are the phase configurations (in the form of block diagrams, fault trees, structure functions, or complete lists of minimal cut sets or minimal path sets), and estimates of the component conditional phase reliabilities π_{kj}, $k = 1,\ldots,n$, $j = 1,\ldots,m$. If one is willing to assume that components have constant failure rates throughout each phase, then the component conditional phase reliabilities are given by

$$\pi_{kj} = e^{-r_{kj}d_j},$$

where r_{kj} is the <u>failure rate</u> of component C_k in phase j, and d_j is the <u>duration</u> of phase j, $k = 1,\ldots,n$, $j = 1,\ldots,m$.

Algorithm for Computing ρ_{PLB-CC}

(1) For $j = 1,\ldots,m$, find the minimal cut sets for the configuration of phase j.

(2) Perform cut cancellation according to the rule given in Section 4. For $j = 1,\ldots,m$, denote the number of minimal cut sets remaining in phase j by $k(j)$, and the $i^{\underline{th}}$ minimal cut set remaining in that phase by K_{ji}, $i = 1,\ldots k(j)$.

(3) For $k = 1,\ldots,n$, compute ρ_{kj} for all $j = 1,\ldots,m$ for which $C_k \varepsilon K_{ji}$ for some $i = 1,\ldots,k(j)$, from

$$\rho_{kj} = \prod_{i=1}^{j} \pi_{ki}.$$

(4) Compute ρ_{PLB-CC} from

$$\rho_{PLB-CC} = \prod_{j=1}^{m} \prod_{i=1}^{k(j)} [1 - \prod_{C_k \varepsilon K_{ji}} (1-\rho_{kj})].$$

The notation necessary to formulate this algorithm in precise mathematical terms obscures its basically very simple content. We can restate it in the following more intelligible form:

(1) Find the minimal cut sets for all phase configurations.

(2) Perform cut cancellation.

(3) Compute ρ_{kj} for each phase j in which component C_k is relevant from

$$\rho_{kj} = \prod_{i=1}^{j} \pi_{ki}$$

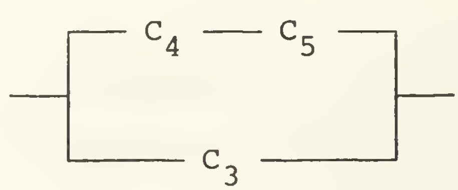
(4) Obtain the "best" lower bound on mission reliability by computing

$$\prod_{\{\text{all phases}\}} \; \prod_{\substack{\{\text{all min cut sets} \\ \text{in each phase}\}}} \left[1 - \prod_{\substack{\{\text{all components in} \\ \text{each min cut set}\}}} (1-\rho_{kj})\right]$$

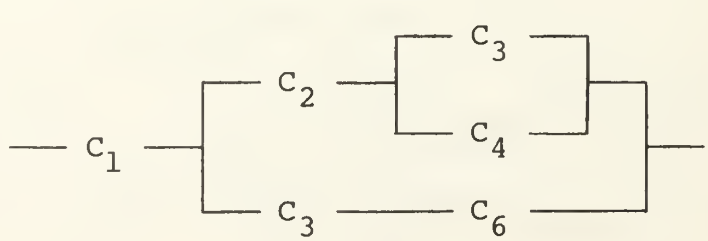
The following example, adapted from [4], illustrates how the algorithm works.

<u>Example 4</u>. A system with six components is to perform a three-phased mission. The phase configurations are represented by the block diagrams

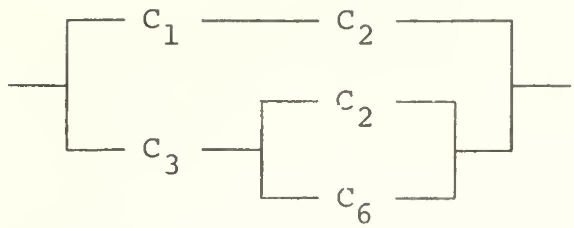
phase 1



phase 2



phase 3



The duration of the phases are $d_1 = 30$ min, $d_2 = 2$ hours, and $d_3 = 10$ hours. It is assumed that the components have failure rates r_{kj} which are constant throughout each phase; estimates of their values (in hours^{-1}) are

j \ k	1	2	3	4	5	6
1	0.000	0.001	0.020	0.040	0.100	0.000
2	0.020	0.003	0.006	0.010	0.500	0.020
3	0.010	0.002	0.005	0.020	0.500	0.020

A lower bound on mission reliability is wanted.

The application of the algorithm yields the following results:

(1) The minimal cut sets are

- in phase 1: #$\{C_3, C_4\}$#, $\{C_3, C_5\}$
- in phase 2: $\{C_1\}$, #$\{C_2, C_3\}$#, #$\{C_2, C_6\}$#, $\{C_3, C_4\}$
- in phase 3: $\{C_1, C_3\}$, $\{C_2, C_3\}$, $\{C_2, C_6\}$

(2) The minimal cut sets marked #{ }# above are cancelled. The remaining minimal cut sets are

- in phase 1: $\{C_3, C_5\}$
- in phase 2: $\{C_1\}$, $\{C_3, C_4\}$
- in phase 3: $\{C_1, C_3\}$, $\{C_2, C_3\}$, $\{C_2, C_6\}$

(3) We have to compute ρ_{12}; ρ_{13}; ρ_{23}; ρ_{31}; ρ_{32}; ρ_{33}; ρ_{42}; ρ_{51}; and ρ_{63}. Since in the present case $\pi_{kj} = e^{-r_{kj}d_j}$, we use the equation

$$\rho_{kj} = \prod_{i=1}^{j} \pi_{ki} = \prod_{i=1}^{j} e^{-r_{ki}d_i} = e^{-\sum_{i=1}^{j} r_{ki}d_i}$$

and obtain the following values for ρ_{kj} (rounded to four decimals)

j\k	1	2	3	4	5	6
1			.9900		.9512	
2	.9608		.9782	.9608		
3	.8694	.9738	.9305			.7866

(4) The bound ρ_{PLB-CC} is given by

$$\rho_{PLB-CC} = [1-(1-\rho_{31})(1-\rho_{51})] \times [1-(1-\rho_{12})]$$

$$\times [1-(1-\rho_{32})(1-\rho_{42})] \times [1-(1-\rho_{13})(1-\rho_{33})]$$

$$\times [1-(1-\rho_{23})(1-\rho_{33})] \times [1-(1-\rho_{23})(1-\rho_{63})].$$

For the values of ρ_{kj} computed in Step (3), we obtain, rounded to four decimals,

$$\rho_{PLB-CC} = .9438.$$

As a comparison, the reliability function for the mission is

$$h = \rho_{12}\,\rho_{33}\,(\rho_{23}+\rho_{63}-\rho_{23}\,\rho_{63})$$

$$+ \rho_{13}\,\rho_{23}\,[(1-\rho_{31})\,\rho_{42}\,\rho_{51} + (\rho_{31}-\rho_{32})\,\rho_{42} + (\rho_{32}-\rho_{33})],$$

and thus the exact mission reliability, rounded to four decimals, is

$$\rho = .9468. \qquad \square$$

7. AN APPROXIMATE HAZARD TRANSFORM FOR PHASED MISSIONS

Recently, Esary and Hayne [1] extended the scope of application of a simple reliability calculus of Rubinstein [6, 7] to coherent systems. This calculus uses an approximate hazard transform and leads to conservative approximations to system reliability. We will show here that its scope can be further extended to phased missions.

The hazard transform of a system with reliability function $h(p_1,\ldots,p_n)$ is defined as

$$H(u_1,\ldots,u_n) = -\log h(e^{-u_1},\ldots,e^{-u_n}),$$

where $u_k = -\log p_k$ is the component hazard of component C_k having reliability p_k, $k = 1,\ldots,n$. The approximate hazard transform H^* considered in [1] can be defined by the following rules:

(1) For a system consisting of a single component C_k,

$$H^* = u_k.$$

(2) For a system which is a combination of two modules (subsystems with disjoint sets of components) having approximate hazard transforms H_1^* and H_2^*,

$H^* = H_1^* + H_2^*$ if the combination is series

$H^* = H_1^* H_2^*$ if the combination is parallel.

(3) For a coherent system with minimal cut sets $K_1,\ldots,K_\ell$ whose approximate hazard transforms are $H_1^*,\ldots,H_\ell^*$,

$$H^* = H_1^* + \ldots + H_\ell^*.$$

It has been shown [1, Theorem 2.5] that this approximate hazard transform is conservative, i.e. indicates greater system hazard (less system reliability) than the exact hazard transform.

In the case of a phased mission, we can go one step further and define an <u>approximate mission hazard transform</u> by the rule

(4) For a phased mission whose simplified phase configurations have approximate hazard transforms $H_1^*,\ldots,H_m^*$, the approximate mission hazard transform is

$$H^* = H_1^* + \ldots + H_m^*,$$

where the component hazards are $u_{kj} = -\log \rho_{kj}$, $k = 1,\ldots n$, $j = 1,\ldots,m$.

We will denote the reliability function corresponding to this approximate mission hazard transform by h^*, i.e.

$$h^* = e^{-H^*}. \tag{23}$$

By comparing steps (1), (2), and (3) of the rule above with the method of computing the minimal cut lower bounds for the reliability of the simplified phase configurations, we can conclude immediately that $e^{-H_j^*} \leq \bar{h}_{LBj}$, $j = 1,\ldots,m$. It then follows from (16) and (23) that

$$h^* \leq \rho_{PLB\text{-}CC}, \tag{24}$$

and hence from (18) and (21) that h^* is a lower bound on mission reliability or, equivalently, that the approximate mission hazard transform is conservative.

An algorithm for computing the lower bound h^* follows the first three steps of the algorithm for computing ρ_{PLB-CC}. The next steps are

(5) Compute the component hazards

$$u_{kj} = -\log \rho_{kj}$$

for all (i,j) for which ρ_{kj} has been computed in Step (3).

(6) Compute the approximate mission hazard transform

$$H^* = \sum_{j=1}^{m} \sum_{i=1}^{k(j)} \prod_{C_k \varepsilon K_{ji}} u_{kj}.$$

(7) Compute the lower bound

$$h^* = e^{-H^*}.$$

A comparison of this algorithm with the one presented in Section 6 indicates that the computation of the lower bound h^* requires more effort than the computation of the lower bound ρ_{PLB-CC}; we also know from (24) that h^* is less precise than ρ_{PLB-CC}. Thus, it may seem counterproductive to pursue the approximate mission transform any further. However, if one is willing to--or has to, for lack of better information--assume constant component phase failure rates, then the component hazards u_{kj} take on the simple form $u_{kj} = \sum_{i=1}^{j} r_{ki} d_i$, and computations are simplified considerably. In this case, an algorithm for h^*

consists of the following steps (expressed in an "intelligible" form):

Algorithm for Computing h^* in the Case of Constant Component Phase Failure Rates.

(1) Find the minimal cut sets for all phase configurations.

(2) Perform cut cancellation.

(3) Compute the component hazard u_{kj} for each phase j in which component C_k is relavent from

$$U_{kj} = \sum_{i=1}^{j} r_{ki} d_i .$$

(4) Obtain the approximate mission hazard transform from

$$H^* = \sum_{\{\text{all phases}\}} \sum_{\substack{\{\text{all min cut sets} \\ \text{in each phase}\}}} \prod_{\substack{\{\text{all components in} \\ \text{each min cut set}\}}} u_{kj}$$

(5) Compute the lower bound h^* from

$$h^* = e^{-H^*} .$$

When component phase failure rates are assumed constant, the approximate mission hazard transform becomes a polynomial in each of the phase durations. Thus, it is well suited for parametric studies, as is demonstrated in the next example.

Example 5. Consider the mission of Example 4. Assume that--all other data being the same as before--the duration of phase 2, d_2, is now uncertain, and that a sensitivity analysis on it is desired.

From the algorithm above, we obtain the following general expression for the approximate mission hazard transform:

$$(25) \quad H^{*} = r_{31}d_1r_{51}d_1$$
$$+ (r_{11}d_1+r_{12}d_2) + (r_{31}d_1+r_{32}d_2)(r_{41}d_1+r_{42}d_2)$$
$$+ (r_{11}d_1+r_{12}d_2+r_{13}d_3)(r_{31}d_1+r_{32}d_2+r_{33}d_3)$$
$$+ (r_{21}d_1+r_{22}d_2+r_{23}d_3)(r_{31}d_1+r_{32}d_2+r_{33}d_3)$$
$$+ (r_{21}d_1+r_{22}d_2+r_{23}d_3)(r_{61}d_1+r_{62}d_2+r_{63}d_3).$$

H^{*} as a function of d_2 can be written as

$$H^{*}(d_2) = a + bd_2 + cd_2^2,$$

with

$$a = d_1r_{11}$$
$$+ d_1^2(r_{11}r_{31}+r_{21}r_{31}+r_{21}r_{61}+r_{31}r_{41}+r_{31}r_{51})$$
$$+ d_1d_3(r_{11}r_{33}+r_{13}r_{31}+r_{21}r_{33}+r_{23}r_{31}+r_{21}r_{63}+r_{23}r_{61})$$
$$+ d_3^2(r_{13}r_{33}+r_{23}r_{33}+r_{23}r_{63}),$$

$$b = r_{12}$$
$$+ d_1(r_{11}r_{32}+r_{12}r_{31}+r_{21}r_{32}+r_{21}r_{62}$$
$$+r_{22}r_{31}+r_{22}r_{61}+r_{31}r_{42}+r_{32}r_{41})$$
$$+ d_3(r_{12}r_{33}+r_{13}r_{32}+r_{22}r_{33}+r_{22}r_{63}+r_{23}r_{32}+r_{32}r_{62}),$$

$$c = r_{12}r_{32} + r_{22}r_{32} + r_{22}r_{62} + r_{32}r_{42}.$$

For the data given in Example 4, the numerical values of these coefficients are

$$a = 0.012030$$

$$b = 0.023333 \text{ hours}^{-1}$$

$$c = 0.000258 \text{ hours}^{-2}.$$

For various durations of phase 2 (in hours), the approximate mission hazard transform H^* and the lower bound on mission reliability h^*, both rounded to four decimals, are shown below.

d_2	H^*	h^*
0	0.0120	0.9880
1	0.0356	0.9650
2	0.0597	0.9420
3	0.0844	0.9191
4	0.1095	0.8963
5	0.1351	0.8736
6	0.1613	0.8510
7	0.1880	0.8286
8	0.2152	0.8064
9	0.2429	0.7843
10	0.2712	0.7625

For $d_2 = 2$ hours, ρ_{PLB-CC} and p have been computed in Example 4. Their values are repeated below, together with the value of h^* $(d_2 = 2 \text{ hours})$, to facilitate a comparison.

$$p = .9468$$

$$\rho_{PLB-CC} = .9438$$

$$h^*(d_2 = 2 \text{ hours}) = .9420.$$

□

In the case of constant component phase failure rates, the approximate hazard transform can also be used to estimate mission reliability when phase durations vary randomly. If $D_1, \ldots, D_m$ are nonnegative random variables denoting the durations of the phases, then the approximate mission hazard transform is $EH^*(D_1, \ldots, D_m)$, where the function H^* is defined as before and E denotes expectation. As an approximation to mission reliability we now use

$$(26) \qquad g^* = e^{-EH^*(D_1, \ldots, D_m)},$$

which is much easier to calculate than the exact value $Ee^{-H^*(D_1, \ldots, D_m)}$. Since e^{-x} is a convex function of x, it follows from Jensen's inequality that $e^{-EH^*(D_1, \ldots, D_m)} \leq Ee^{-H^*(D_1, \ldots, D_m)}$, and therefore g^* is a lower bound on mission reliability.

In our last example, we show how this approximation can be used, even without a complete knowledge of the probability distributions of the D_j's.

Example 6. Consider again the mission of Example 4, but this time assume that--all other data being the same as before--the durations of phases 2 and 3 are random. The mean durations are known to be $ED_2 = d_2 = 2$ hours and $ED_3 = d_3 = 10$ hours, and the total duration of these two phases together is $D_2 + D_3 = 12$ hours. An estimate for the mission reliability under these circumstances is wanted.

By rearranging the terms of (25) we can express H^* as a function of D_2 and D_3 by

$$H^*(D_2,D_3) = a_1 + a_2D_2 + a_3D_3 + a_4D_2^2 + a_5D_3^2 + a_6D_2D_3,$$

where the coefficients $a_1,\ldots,a_6$ depend only on the known duration of phase 1 and the component phase failure rates. Since $D_2 + D_3$ = constant, then $\mathrm{Var}\ D_2 = \mathrm{Var}\ D_3$, and $\mathrm{Cov}(D_2,D_3) = -\mathrm{Var}\ D_2 = -\mathrm{Var}\ D_3$. Denoting this common but unknown variance by σ^2, we can write $ED_2^2 = \sigma^2 + d_2^2$, $ED_3^2 = \sigma^2 + d_3^2$, and $ED_2D_3 = d_2d_2 - \sigma^2$, and obtain

$$EH^*(D_2,D_3) = a_1 + a_2d_2 + a_3d_3 + a_4d_2^2 + a_5d_3^2 + a_6d_2d_3 + \sigma^2(a_4+a_5-a_6),$$

or, numerically,

$$EH^*(D_2,D_3) = 0.059728 + 0.000071\ \sigma^2/\text{hours}^2.$$

For $\sigma^2 = 0$, i.e. when the durations of phases 2 and 3 take on their expected values with probability one, $EH^*(D_2,D_3) = 0.0597$ and $g^* = .9420$, which agrees with the corresponding results of Example 5. As σ^2 increases, $EH^*(D_2,D_3)$ increases and g^* decreases: since σ^2 cannot be greater than 20 hours^2 under the given conditions, the maximum value of $EH^*(D_2,D_3)$ is 0.0611, and the corresponding minimum value of g^* is .9407. We can therefore conclude that the mission reliability is at least 0.94.

□

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